Special Functions

1-Gamma function :

The gamma function is defined by integral

$$\Gamma\alpha = \int_0^\infty x^{\alpha-1}e^{-x}\ dx \quad , \qquad \alpha > 0$$

$$\Gamma 1 = \int_0^\infty x^{1-1}e^{-x}\ dx = \int_0^\infty e^{-x}dx = -[e^{-x}]_0^\infty = -[e^{-\infty} - e^0] = 1$$

Note

1) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \alpha > 0$
2) $\Gamma(n) = (n-1)! \qquad n = 1,2,3,........$
3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Examples :

1) $\Gamma\frac{7}{2} = \Gamma(\frac{5}{2}+1) = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}.\frac{3}{2}\Gamma(\frac{3}{2}) = \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2}) \quad = \frac{15}{8}\sqrt{\pi}$
2) $\Gamma(\frac{10}{3}) = \frac{7}{3}\Gamma(\frac{7}{3}) = \frac{7}{3}.\frac{4}{3}\Gamma(\frac{4}{3}) = \frac{7}{3}.\frac{4}{3}.\frac{1}{3}\Gamma(\frac{1}{3}) = \frac{28}{27}\Gamma(\frac{1}{3})$
3) $\Gamma(6) = (6-1)! = 5! = 5.4.3.2.1 = 120$
4) $\Gamma(4) = (4-1)! = 3! = 3.2.1 = 6$
5) $\Gamma(1) \ = (1-1)! = 0! = 1$

Examples: Express the following integration by using of gamma functions and find values .

1) $\int_0^\infty x^{-\frac{1}{2}}\ e^{-x}dx$ 2) $\int_0^\infty x^{\frac{2}{3}}\ e^{-x}dx$

3) $\int_0^\infty \sqrt{x}\, e^{-x}dx$ 4) $\int_0^\infty x^4 e^{-x}dx$

5) $\int \sqrt[4]{x}\, e^{-\sqrt{x}}dx$

Solution:

$$1)\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx=\int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx=\Gamma(\frac{1}{2})=\sqrt{\pi}$$

$$4)\int_0^\infty x^4e^{-x}dx=\int_0^\infty x^{5-1}e^{-x}dx=\Gamma(5)=4!=4.3.2.1=24$$

$$5)\int_0^\infty \sqrt[4]{x}\,e^{-\sqrt{x}}dx=\int_0^\infty x^{\frac{1}{4}}e^{-x^{\frac{1}{2}}}dx$$

$Let\quad y=\sqrt{x}\quad \Rightarrow x=y^2\quad \Rightarrow dx=2ydy$

$When\quad x=0\Rightarrow y=0\qquad and\quad x=\infty\Rightarrow y=\infty$

$$\therefore\int_0^\infty \sqrt[4]{x}\,e^{-\sqrt{x}}dx=\int_0^\infty \sqrt[4]{y^2}\,e^{-y}.2y\,dy=2\int_0^\infty y^{\frac{3}{2}}e^{-y}dy=2\Gamma(\frac{5}{2})=2.\frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2})=\frac{3\sqrt{\pi}}{2}$$

Exercises : find the value by using gamma function of the following integration

$1)\int_0^\infty x^2e^{-x^2}dx \qquad 2)\int_0^\infty x\,e^{-x^3}dx$

$3)\int_0^\infty e^{-x^3}dx \qquad 4)\int_0^\infty x^2\,e^{-3x}dx$

Note: $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$

Example:

$$\int_0^2\frac{dx}{\sqrt{\ln(\frac{2}{x})}}$$

$Let\ y=\ln(\frac{2}{x})\Rightarrow\frac{2}{x}=e^y\Rightarrow x=2e^{-y}\Rightarrow dx=-2e^{-y}dy$

$When\ x=0\Rightarrow y=\ln\frac{2}{0}=\ln(\infty)=\infty$

$When\ x=2\Rightarrow y=\ln\frac{2}{2}=\ln 1=0$

$$\therefore\int_0^2\frac{dx}{\sqrt{\ln(\frac{2}{x})}}=\int_\infty^0\frac{-2e^{-y}}{\sqrt{y}}dy=-2\int_\infty^0 y^{-\frac{1}{2}}e^{-y}dy=2\int_0^\infty y^{\frac{1}{2}-1}e^{-y}dy=2\Gamma(\frac{1}{2})=2\sqrt{\pi}$$

2- Bate function

The bate function is defined by integral

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m,n>0$$

Note: The related between gamma function and bate .

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Example: Express the integration following of bate and gamma function and find values.

1) $\int_0^1 x^3(1-x)^2\,dx$ 2) $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx$

3) $\int_0^1 \frac{x^4}{\sqrt{1-x^2}}\,dx$ 4) $\int_0^1 x^2(1-x^2)^{\frac{3}{2}}\,dx$

5) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ 6) $\int_0^1 \frac{x^2}{\sqrt{1-x^3}}\,dx$

Solution:

1) $\int_0^1 x^3(1-x)^2\,dx=B(4,3)=\frac{\Gamma(4)\Gamma(3)}{\Gamma(4+3)}=\frac{3!.2!}{6!}=\frac{3.2.1.2.1}{6.5.4.3.2.1}=\frac{1}{60}$

2) $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx=\int x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}\,dx=\int x^{\frac{1}{2}-1}(1-x)^{\frac{3}{2}-1}\,dx=B(\frac{1}{2},\frac{3}{2})=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}+\frac{3}{2})}$

$$=\frac{\sqrt{\pi}.\frac{1}{2}\Gamma(\frac{1}{2})}{\Gamma(2)}=\frac{\sqrt{\pi}.\frac{1}{2}.\sqrt{\pi}}{1!}=\frac{\pi}{2}$$

$$6)\int_0^1 \frac{x^2}{\sqrt{1-x^3}}dx \quad = \int_0^1 x^2(1-x^3)^{\frac{-1}{2}}dx$$

$$Let \quad y = x^3 \Rightarrow x = y^{\frac{1}{3}} \quad \Rightarrow dx = \frac{1}{3}y^{\frac{-2}{3}}dy$$

$$When \quad x=0 \Rightarrow y=0 \quad and \quad x=1 \Rightarrow y=1$$

$$\int_0^1 x^2(1-x^3)^{\frac{-1}{2}}dx = \int_0^1 y^{\frac{2}{3}}(1-y)^{\frac{-1}{2}}.\frac{1}{3}y^{\frac{-2}{3}}dy = \frac{1}{3}\int_0^1 (1-y)^{\frac{-1}{2}}dy = \frac{1}{3}B(1,\frac{1}{2})$$

$$= \frac{\Gamma(1)\Gamma(\frac{1}{2})}{3\Gamma(1+\frac{1}{2})} = \frac{0!.\sqrt{\pi}}{3.\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{\sqrt{\pi}}{\frac{3}{2}\sqrt{\pi}} = \frac{2}{3}$$

Error function

Defined error function by the following:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$erf(\infty) = 1$ $\quad$ *Since*

$$erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt$$

$$Let \quad y = t^2 \Rightarrow t = y^{\frac{1}{2}} \Rightarrow dt = \frac{1}{2}y^{-\frac{1}{2}}dy$$

$$erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-y}.\frac{1}{2}y^{-\frac{1}{2}}dy = \frac{1}{\sqrt{\pi}}\int_0^\infty y^{-\frac{1}{2}}e^{-y}dy = \frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

Fourier series

Definition:

Let $f(x)$ defined function on interval $(-L.L)$ then factorial Fourier series of function $f(x)$ is :

$$f(x) = a_\circ + \sum_{n=1}^{\infty}[a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})]$$

Where $a_\circ, a_n$ *and* b_n are the Fourier coefficients

$$a_\circ = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx$$

$$b_n = \frac{1}{L}\int f(x)\sin(\frac{n\pi x}{L})\,dx$$

Definition:

A function $y = f(x)$ is said to be **even** if $f(-x) = f(x)$ for all values of x
A function $y = f(x)$ is said to be **odd** if $f(-x) = -f(x)$ for all values of x

Examples :

1) $f(x) = x^2 \Rightarrow f$ *is even since*
$f(-x) = (-x)^2 = x^2 = f(x)$

2) $f(x) = x^3 \Rightarrow f$ *is odd since*
$f(-x) = (-x)^3 = -x^2 = -f(x)$

Notes:

1) if for all $f(x), g(x)$ are even functions then $f(x) \pm g(x)$, *and* $f(x).g(x)$ even functions .
2) if for all $f(x), g(x)$ are odd functions then $f(x) \pm g(x)$ odd function and $f(x).g(x)$ even function.
3) If $f(x)$ odd function and $g(x)$ even function then $f(x) \pm g(x)$ not even and not odd , and $f(x).g(x)$ odd function.

Example:

1) $f(x) = x^2 + x \quad \to f$ *function is not odd and not even*
2) $f(x) = x\cos x \quad \to \quad f$ *is odd function*
3) $f(x) = x^3 \sin x \quad \to \quad f$ *is even function*

Notes :

1) if the function $f(x)$ even function defined on interval $(-a, a)$ then :

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

2) if the function $f(x)$ odd function defined on interval $(-a, a)$ then :

$$\int_{-\infty}^{a} f(x)\,dx = 0$$

Notes:

1) When the $f(x)$ is odd function then $a_\circ , a_n = 0$
2) When the $f(x)$ is even function then $b_n = 0$
3) $\sin(n\pi) = 0$ for all value $n = 0, \pm1, \pm2$,...........
4) $\cos(n\pi) = (-1)^n$ for all value $n = 0, \pm1, \pm2$,...........

Example: find Fourier series of function $f(x) = x^2$ and defined on interval $(-\pi , \pi)$

Solution:

Since $f(x) = x^2$ is even function on interval $(-\pi , \pi)$ then $b_n = 0$
We find $a_\circ$, a_n now

$$a_\circ = \frac{1}{2L}\int_{-l}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{2\pi}\int_{0}^{\pi} x^2 dx = \frac{1}{\pi}[\frac{x^3}{3}]_0^\pi = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(\frac{n\pi x}{\pi})dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(nx)\,dx$$

$$Let\ u = x^2 \Rightarrow du = 2xdx \quad , \quad dv = \cos(nx) \Rightarrow v = \frac{\sin(nx)}{n}$$

$$= \frac{2}{\pi}[[\frac{x^2 \sin(nx)}{n}]_0^\pi - \int_0^\pi 2x.\frac{\sin(nx)}{n}dx] = \frac{2}{\pi}[\frac{\pi^2 \sin(n\pi)}{n} - 0] - \frac{4}{\pi n}\int_0^\pi x\sin(nx)dx$$

$$= \frac{-4}{n\pi}\int_0^\pi x\sin(nx)\,dx$$

$$Let\ u = x \Rightarrow du = dx\ , \qquad dv = \sin(nx)dx \Rightarrow v = \frac{-\cos(nx)}{n}$$

$$= -\frac{4}{n\pi}[[-\frac{x\cos(nx)}{n}]_0^\pi - \int_0^\pi -\frac{\cos(nx)}{n}dx]$$

$$= -\frac{4}{n\pi}[-\frac{\pi\cos(n\pi)}{n} - 0 + \frac{1}{n^2}[\sin(nx)]_0^\pi = \frac{-4}{n\pi}[-\frac{\pi\cos(n\pi)}{n} + \frac{1}{n^2}(0-0)]$$

$$= \frac{4}{n^2}\cos(n\pi)$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$$f(x) = a_\circ + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}[\frac{4}{n^2}(-1)^n \cos(\frac{n\pi x}{\pi}) + 0]$$

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

Example: Find Fourier series of function $f(x) = x$ on the interval $(-1,1)$

Solution:

Since the function $f(x) = x$ is odd function on the interval $(-1,1)$. Then $a_\circ , a_n = 0$

$$b_n = \frac{1}{L}\int_{-1}^{1} f(x).\sin(\frac{n\pi x}{L})\, dx = \int_{-1}^{1} x\sin(n\pi x)\, dx$$

$$= 2\int_{0}^{1} x\sin(n\pi x) dx$$

$$Let\ x = u \Rightarrow dx = du \qquad , dv = \sin(n\pi x) \quad \Rightarrow v = -\frac{\cos(n\pi x)}{n\pi}$$

$$= 2[[-\frac{x\cos(n\pi x)}{n\pi}]_0^1 - \int_0^1 -\frac{\cos(n\pi x)}{n\pi} dx]$$

$$= 2[-\frac{\cos(n\pi)}{n\pi} - 0 - [\frac{\sin(n\pi x)}{n^2\pi^2}]_0^1] = 2[-\frac{(-1)^n}{n\pi} - \frac{1}{(n\pi)^2}(\sin(n\pi) - \sin(0)]$$

$$= 2[-\frac{(-1)^n}{n\pi} - 0 - 0] = \frac{-2}{n\pi}(-1)^n$$

$$b_n = \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = a_\circ + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

$$f(x) = 0 + \sum_{n=1}^{\infty}[0 + \frac{2}{n\pi}(-1)^{n+1} \sin(n\pi x)$$

$$f(x) = x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)$$

Example: Find Fourier series of function $f(x) = 2x$ on the interval $(-3,3)$

H.W.

الأخطاء Error

أنواع الأخطاء Type of Error

-1 الخطأ المطلق (Absolute Error): هو الفرق بين القيمة الحقيقية والقيمة التقريبية ويرمز له بالرمز e_x أي إن $e_x = X - \overline{X}$. حيث إن X تمثل القيمة الحقيقية و $\overline{X}$ تمثل القيمة التقريبية .

-2 الخطأ النسبي (Relative Error): وهو الخطأ المطلق مقسوما على القيمة التقريبية ويرمز له بالرمز R_e أي إن $R_e = \dfrac{e_x}{\overline{X}} = \dfrac{X - \overline{X}}{\overline{X}}$

مثال : لتكن 0.0008 قيمة تقريبية للقيمة الحقيقية 0.0009 أوجد الخطأ المطلق والخطأ النسبي .

الحل :

$$X = 0.0009 \qquad \overline{X} = 0.0008$$

$$\therefore e_x = X - \overline{X} = 0.0009 - 0.0008 = 0.0001$$

$$R_e = \frac{e_x}{\overline{X}} = \frac{0.0001}{0.0008} = 0.125$$

تمثيل الأعداد

يمكن تمثيل أي عدد حقيقي X لأي أساس كان بصيغة الفارزة السائبة بالشكل التالي

$$X = f * b^k$$

حيث إن

f **: تمثل الجزء ألكسري (fraction)**

b **: أساس النظام المستخدم**

k **: الأس ويكون عدد صحيح**

مثال : اكتب العدد بصيغة الفارزة السائبة للعدد $X = 731.56$

الحل :

$$X = 0.73156 * 10^3$$

ملاحظات :

-1 إذا كانت حركة الفارزة إلى اليسار نضرب في 10^{+} ويكون الأس بعدد مرات الحركة.

-2 إذا كانت حركة الفارزة إلى اليمين نضرب في 10^{-} وبعدد مرات حركة الفارزة

الجزء ألكسري يحتوي على n من الأرقام الواقعة على يمين الفارزة على الشكل $f = \pm 0.d_1 d_2 .....d_n$ حيث إن $d_1 ....d_n$ هي أرقام

يسمى العدد المكتوب بواسطة الفارزة السائبة بأنة عدد طبيعي إذا كانت $d_1 \neq 0$ وإلا فيكون عدد غير طبيعي حيث يمكن تحويل أي عدد غير طبيعي إلى عدد طبيعي وذلك بحركة الفارزة إلى اليمين

$$X = 0.0024 * 10^5$$
$$X = 0.24 * 10^3$$

في النظام العشري تكون بالصيغة الآتية $\frac{1}{10} \leq f < 1$

وفي النظام الثنائي تكون بالصيغة $\frac{1}{2} \leq f < 1$

وبصورة عامة إذا كان الأساس المستخدم b فان $\frac{1}{b} \leq f < 1$

مثال : إذا كنا نستخدم إعداد في حالة الفارزة السائبة بطول 4 وكانت x^*, y^*, z^* هي كالأتي

$$x^* = 0.6359 * 10^6 \quad , y^* = 0.2180 * 10^{-2} \quad , \quad z^* = -5846 * 10^3$$

فاوجد $x^*.y^*$, $x^* + z^*$

الحل : أولا نقوم بتحويل العدد الذي يحمل القوة الصغرى إلى نفس العدد الثاني وذلك بتحريك كسر العدد الذي يحوي على الأس الأصغر إلى اليمين عدد من المراتب تساوي الفرق بين الأسين

$$z^* = 0.5846 * 10^3 = 0.0005846 * 10^6$$
$$\therefore z^* + x^* = 0.0005846 * 10^6 + 0.6359 * 10^6$$
$$= 0.6364846 * 10^6 = 0.6365 * 10^6$$

$$x^*.y^* = (0.6359 * 10^6).(0.2180 * 10^{-2})$$
$$= 0.138626 * 10^4$$
$$= 0.1386 * 10^4$$

مثال: حول الإعداد العشرية التالية إلى إعداد في حالة الفارزة السائبة الطبيعية بطول 4 ثم اوجد ناتج الجمع .

1) $x_1 = 165.2$, $x_2 = 21.00$

2) $x_1 = 1.2462$, $x_2 = 0.3290 * 10^{-1}$

3) $x_1 = 106.4$, $x_2 = -31.73$

H.W.

أخطاء التدوير Round off Error

يمكن تجزئة أي عدد حقيقي في نظام أساسة 10 كالأتي :

$$x = f_x * 10^k + g_x * 10^{k-n}$$

حيث ان f_x لها n من الأرقام كما وان :-

$$\frac{1}{10} \le |f_x| < 1 \quad , \quad 0 \le |g_x| < 1$$

إن أخطاء التدوير يمكن حسابه بطريقتين .:

1- **خطا التدوير بالبتر (خطا البتر)**.: في هذه الحالة يهمل الجزء الثاني لنحصل على القيمة التقريبية الآتية $\bar{x} = f_x * 10^k$ ويمكن إيجاد حدا أعلى للخطأ المطلق في هذه القيمة كما يلي .:

$$|e_x| = |x - \bar{x}| = |f_x * 10^k + g_x * 10^{k-n} - f_x * 10^k|$$

$$= |g_x * 10^{k-n}| = |g_x| * 10^{k-n}$$

$$\Rightarrow |e_x| = |g_x| * 10^{k-n}$$

$$\Rightarrow |e_x| < 10^{k-n} \quad , (|g_x| < 1)$$

كذلك بالنسبة إلى الخطاء النسبي حيث يكون الحد الأعلى للخطاء النسبي كما يلي :

$$|\mathrm{Re}_x| = \left|\frac{e_x}{\bar{x}}\right| = \frac{|e_x|}{|\bar{x}|} < \frac{10^{k-n}}{|f_x| * 10^k} \le \frac{10 * 10^{k-n}}{10^k} \quad , \quad (|f_x| \ge \frac{1}{10} \Rightarrow \frac{1}{|f_x|} \le 10)$$

$$|\mathrm{Re}_x| < 10^{1-n}$$

2- **خطاء التدوير ألتناسقي (الخطاء المدور)** .: في هذه الحالة القيمة التقريبية إلى x تصبح كالأتي .:

$$\bar{x} = \begin{cases} f_x * 10^k & if \quad |g_x| < \frac{1}{2} \\ f_x * 10^k + 10^{k-n} & if \quad |g_x| \ge \frac{1}{2} \end{cases}$$

السطر الثاني يعني إضافة 1 إلى الرقم الأخير في f_x .
وعلية تكون القيمة المطلقة للخطاء كالأتي :

الحد الأعلى للخطاء المطلق هو $$|e_x| < \frac{1}{2} * 10^{k-n}$$

إما الحد الأعلى للقيمة المطلقة للخطاء النسبي هو $$|\mathrm{Re}_x| < \frac{1}{2} * 10^{1-n}$$

ملاحظة.: بصورة عامة يمكن تجزئة أي عدد حقيقي x في نظام أساسة b كالآتي :

$$x = f_x * b^k + g_x * b^{k-n}$$

حيث إن f_x لها n من الأرقام كما إن $\frac{1}{b} \le |f_x| < 1 \quad , \quad 0 \le |g_x| < 1$

1- في حالة التدوير بالبتر فان $\bar{x} = f_x * b^k$

وعلية تكون الحدود العليا للقيم المطلقة للأخطاء كالآتي .:

$$|e_x| < b^{k-n}$$
$$|\mathrm{Re}_x| < b^{1-n}$$

2- في حالة التدوير ألتناسقي فان .:

$$\bar{x} = \begin{cases} f_x * b^k & if \quad |g_x| < \frac{1}{2} \\ f_x * b^k + b^{k-n} & if \quad |g_x| \ge \frac{1}{2} \end{cases}$$

وعلية تكون الحدود العليا للقيم المطلقة للأخطاء كالآتي .:

$$|e_x| < \frac{1}{2} * b^{k-n}$$
$$|\mathrm{Re}_x| < \frac{1}{2} * b^{1-n}$$

مثال .: احسب خطأ البتر والخطاء المدور (المطلق والنسبي) للعدد $x = 732.48261$ **عندما** n**=4**

الحل .:

$$x = 0.73248261 * 10^3$$
$$= 0.7324 * 10^3 + 0.8261 * 10^{-1}$$

أولا: نجد خطا البتر

$$\bar{x} = 0.7324 * 10^3$$
$$|e_x| = |x - \bar{x}|$$
$$= |0.7324 * 10^3 + 0.8241 * 10^{-1}| - 0.7324 * 10^3$$
$$|e_x| = 0.8261 * 10^{-1}$$

$$|\mathrm{Re}_x| = \frac{|e_x|}{|\bar{x}|} = \left|\frac{0.8261 * 10^{-1}}{0.7324 * 10^3}\right| \approx 1.1 * 10^{-4}$$

باسم خضير

ثانيا: نجد خطا المدور

$$\bar{x} = 0.7324*10^3 + 10^{-1}$$
$$= 0.7324*10^3 + 0.0001*10^3$$
$$= 0.7325*10^3$$

$$|e_x| = |1-g_x|*10^{k-n} = |1-0.8261|*10^{3-4} = 0.1739*10^{-1}$$

$$|\mathrm{Re}_x| = \left|\frac{e_x}{\bar{x}}\right| = \left|\frac{0.1739*10^{-1}}{0.7325*10^3}\right| \approx 0.24*10^{-4}$$

مثال .: اوجد الخطاء النسبي للعدد **H.W.** $x = 0.7324*10^3 + 0.8261*10^{-1}$

مثال .: اوجد كل مما يأتي :

1- **الحدود العليا للخطأ المدور (المطلق والنسبي)**
2- **الحدود العليا لخطا البتر (المطلق والنسبي)**

للعدد $x = 0.7324*10^3 + 0.8261*10^{-1}$

الحل:

-1

$$|e_x| < \frac{1}{2}*10^{k-n} = \frac{1}{2}*10^{3-4} = 0.5*10^{-1}$$
$$|e_x| < 0.05$$
$$|\mathrm{Re}_x| < \frac{1}{2}*10^{1-n} = 0.5*10^{1-4} = 0.5*10^{-3}$$
$$|\mathrm{Re}_x| < 5*10^{-4}$$

-2

$$|e_x| < 10^{k-n} = 10^{3-4} = 10^{-1} = 0.1$$
$$|\mathrm{Re}_x| < 10^{1-n} = 10^{1-4} = 10^{-3} = 0.001$$

تمارين : واجب

1- **احسب مقدار الخطأ المدور وخطا البتر للإعداد الآتية حسب قيمة** n **المعطاة :**

Number	**n**
546.25454	**4**
5.46254	**3**
0.00372	**3**
67843.27815	**4**
1.269	**1**
1.00269	**2**

2- **(a) اوجد الحدود العليا للخطأ المدور (المطلق والنسبي) للإعداد اعلاة**
(b) اوجد الحدود العليا لخطا البتر (المطلق والنسبي) للإعداد اعلاة

تأثير أخطاء التدوير على العمليات الحسابية

لتكن $\bar{x}\ ,\ \bar{y}$ قيمتين تقريبيتين للعددين $x\ ,y$ بخطأ مطلق e_x و e_y وخطأ نسبي Re_x و Re_x على التوالي فان :

1- عملية الجمع

$$e_{x+y}=e_x+e_y$$

$$\mathrm{Re}_{x+y}=\frac{\bar{x}}{\bar{x}+\bar{y}}\mathrm{Re}_x+\frac{\bar{y}}{\bar{x}+\bar{y}}\mathrm{Re}_y$$

2- عملية الطرح

$$e_{x-y}=e_x-e_y$$

$$\mathrm{Re}_{x-y}=\frac{\bar{x}}{\bar{x}-\bar{y}}\mathrm{Re}_x-\frac{\bar{y}}{\bar{x}-\bar{y}}\mathrm{Re}_y$$

3- عملية الضرب

$$e_{x.y}=\bar{x}.e_y+\bar{y}.e_x$$

$$\mathrm{Re}_{x.y}=\mathrm{Re}_x\ \mathrm{Re}_y$$

4- عملية القسمة

$$e_{\frac{x}{y}}=\frac{e_x}{\bar{y}}-\frac{\bar{x}.e_y}{\bar{y}^2}$$

$$\mathrm{Re}_{\frac{x}{y}}=\mathrm{Re}_x-\mathrm{Re}_y$$

مثال :. لتكن كل من $\bar{x}=62.45$ و $\bar{y}=13.2$ أعداد مدورة .جد الحدود العليا للخطأ المطلق والنسبي لكل من $x\ ,\ y\ ,\ x.y\ ,\ x-y$

الحل :. نقوم بتحويل الإعداد إلى حالة الفارزة السائبة الطبيعية

$$\bar{x} = 62.45 = 0.6245 * 10^{2} \Rightarrow n = 4 \,,\, k = 2$$

$$\bar{y} = 13.2 = 0.132 * 10^{2}$$

$$|e_x| < \frac{1}{2} * 10^{k-n} = \frac{1}{2} * 10^{2-4} = 0.5 * 10^{-2} = 0.005$$

$$|e_y| < \frac{1}{2} * 10^{k-n} = 0.5 * 10^{2-3} = 0.5 * 10^{-1} = 0.05$$

$$|\mathrm{Re}_x| < \frac{1}{2} * 10^{1-n} = 0.5 * 10^{1-4} = 0.5 * 10^{-3} = 0.0005$$

$$|\mathrm{Re}_y| < \frac{1}{2} * 10^{1-n} = 0.5 * 10^{1-3} = 0.5 * 10^{-2} = 0.005$$

$$|e_{x.y}| = |\bar{x}.e_y + \bar{y}.e_x| \le |\bar{x}.e_y| + |\bar{y}.e_x|$$

$$= 62.45 * 0.05 + 13.2 * 0.005$$

$$= 3.1225 + 0.0660$$

$$= 3.1885$$

$$|e_{x-y}| = |e_x - e_y| \le |e_x| + |e_y|$$

$$= 0.005 + 0.05$$

$$= 0.055$$

$$|\mathrm{Re}_x| < \frac{1}{2} * 10^{1-n} = 0.5 * 10^{1-4} = 0.5 * 10^{-3} = 0.0005$$

$$|\mathrm{Re}_y| < \frac{1}{2} * 10^{1-n} = 0.5 * 10^{1-3} = 0.5 * 10^{-2} = 0.005$$

$$|\mathrm{Re}_{x.y}| = |\mathrm{Re}_x + \mathrm{Re}_y| \le |\mathrm{Re}_x| + |\mathrm{Re}_y| = 0.0005 + 0.005 = 0.0055$$

$$|\mathrm{Re}_{x-y}| = \left| \frac{\bar{x}}{\bar{x} - \bar{y}} \mathrm{Re}_x - \frac{\bar{y}}{\bar{x} - \bar{y}} \mathrm{Re}_y \right|$$

$$\le \left| \frac{\bar{x}}{\bar{x} - \bar{y}} \mathrm{Re}_x \right| + \left| \frac{\bar{y}}{\bar{x} - \bar{y}} \mathrm{Re}_y \right|$$

$$= \frac{62.45}{62.45 - 13.2} * 0.0005 + \frac{13.2}{62.45 - 13.2} * 0.005$$

$$= 0.0063401 + 0.0013401$$

$$= 0.0076802$$

باسم خضر

Advance Mathematics

Basim Khdaer

Al-Mustansiriya / Computer Science Dept.

2008-2009

Differential Equations

A differential equation is a relation between the independent, dependent variables and their differential coefficients.

Note: the derivative by defined $y = f(x)$ with respect to x

$$\frac{dy}{dx} = f'(x) = \frac{df(x)}{dx}$$

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d^2f(x)}{dx^2}$$

Types Of Differential Equations And Definitions

Ordinary Differential Equations

An Ordinary Differential Equation is a differential equation that depends on only one independent variable.

Example:

1) $y'' + 3y = x^2$ 2) $(2x + 2y)^2 y' = 1$

3) $(1 + 2y'^2)^{\frac{3}{2}} = 8y''$ 4) $udu + xdx + zdz$

4) $x^2 + 2xyy' + (yy')^2 = (zz')^2$

Partial Differential Equations

A Partial Differential Equation is differential equation in which the dependent variable depends on two or more independent variables.

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Example:

1) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ 2) $\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=0$

-

3) $(\frac{\partial^2 z}{\partial x^2})(\frac{\partial^2 z}{\partial y^2})(\frac{\partial^2 z}{\partial x\partial y})=0$

Order of Differential Equation:

The order of a differential is the order of the highest derivative entering the equation.

Example:

1) $(3x+2y)^2 y'=1$

2) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

3) $udu+xdu+zdu$

4) $x^2++2xyy'+(yy')^2=(zz')^2$

} is called one-order differential equation

1) $y''+3y=x^2$

2) $(1+2y'^2)^{\frac{3}{2}}=8y''$

3) $\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=0$

4) $(\frac{\partial^2 z}{\partial x^2})(\frac{\partial^2 z}{\partial y^2})(\frac{\partial^2 z}{\partial x\partial y})=0$

} is called a second-order differential equation

Degree of a Differential Equation: The degree of a differential equation is the highest power of the highest order derivative after making the equation free from radicals and fractional indices as far as the derivatives are concerned.

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Example: find degree of the differential $\sqrt[3]{(y'')^2} = \sqrt{1+(y')^2}$

Solution:

$$\sqrt[3]{(y'')^2} = \sqrt{1+(y')^2} \Rightarrow (y'')^2 = [1+(y')^2]^{\frac{3}{2}}$$

$$\Rightarrow (y'')^4 = [1+(y')]^3$$

Then the degree of differential equation is 4

The differential equation $\left.\begin{matrix} y'' + 3y = x^2 \\ (3x+2y)^2 y' = 1 \\ x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z \end{matrix}\right\}$ are one degree

And the equations $\left.\begin{matrix} [1+2y'^2]^{\frac{3}{2}} = 8y'' \\ [\dfrac{\partial^2 z}{\partial x^2}][\dfrac{\partial^2 z}{\partial y^2}][\dfrac{\partial^2 z}{\partial x \partial y}]^2 = 0 \\ x^2 + 2xyy' + (yy')^2 = (zz')^2 \end{matrix}\right\}$ are tow degree

Linear differential equations

A linear differential equation is a differential equation of the form

$$a_{\circ}(x)y^{(n)} + a_1(x)y^{(n-1)} + ...+ a_{n-1}(x)y' + a_n(x)y = f(x)................ \quad (1)$$

Where $a_{\circ}(x), a_1(x), ..., a_n(x), f(x)$ are function defined with x on the interval $a \le x \le b$

If the $f(x) = 0$ then function (1) is called a homogeneous equation

$$a_{\circ}(x)y^{(n)} + a_1(x)y^{(n-1)} + ...+ a_{n-1}(x)y' + a_n(x)y = 0 \quad .... \quad (2)$$

And if $a_{\circ}(x), a_1(x), ..., a_n(x)$ are constant coefficients the equation (2) is called linear homogeneous equation .

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Example:

$xy''+2xy'+y=9$ linear differential equation non homogeneous

$y'''+2y''+y'+5y=0$ linear differential equation homogeneous of constant coefficients

Solution of a Differential Equation: The functional relation-ship between the independent variable and the dependent variable (such as y = f(x)) which satisfies the given differential equation is called the solution of the differential equation.

Example:

Is $y=x\ln(x)-x$ solution of the differential equation $x\frac{dy}{dx}=x+y$

Solution:

$$y=x\ln(x)-x \quad \Rightarrow \frac{dy}{dx}=x.\frac{1}{x}+\ln x-1$$

$$\frac{dy}{dx}=\ln x$$

Substation y',y in the above differential equation

$$x\frac{dy}{dx}=x+y \Rightarrow x\ln x = x+x\ln x - x \Rightarrow x\ln x = x\ln x$$

$y=x\ln(x)-x$ Is solution of differential equation

Example:

Is $y=x^2$ solution of the differential equation $y\frac{dy}{dx}+x=0$

Solution:

$$y=x^2 \ \Rightarrow \ \frac{dy}{dx}=2x$$

Substation y',y in the above differential equation

$$y\frac{dy}{dx}+x=0 \Rightarrow x^2.2x+x=2x^3+x\neq 0$$

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i.e. $y = x^2$ is not solution of the differential equation $y\frac{dy}{dx} + x = 0$

Example:

Is $x^2 + y^2 = 0$ solution of the differential equation $yy' + x = 0$ H.W.

Example:

Is $y = 2e^{-x} + 3e^{-2x}$ solution of the differential equation $y'' + 3y' + 2y = 0$

Solution:

$$y = 2e^{-x} + 3e^{-2x} \Rightarrow y' = -2e^{-x} - 6e^{-2x}$$
$$y'' = 2e^{-x} + 12e^{-2x}$$

Substation y'', y', y in the above differential equation

$$y'' + 3y' + 2y = 0 \Rightarrow 2e^{-x} + 12e^{-2x} + 3(-2e^{-x} - 6e^{-2x}) + 2(2e^{-x} 3e^{-2x})$$
$$\Rightarrow \quad (2e^{-x} - 6e^{-x} + 4e^{-x}) + (12e^{-2x} - 18e^{-2x} + 6e^{-2x}) = 0 + 0 = 0$$

i.e. $y = 2e^{-x} + 3e^{-2x}$ is solution of the differential equation $y'' + 3y' + 2y = 0$

General solution of a differential equation: If the solution of a differential equation of order n contains n arbitrary constants, then it is called the General solution of the differential equation

Particular solution of a differential equation: A solution obtained, by assigning particular values to the arbitrary constants in the general solution of the differential equation, is called its particular solution.

Example: find the general solution and particular solution of the differential equation $y'' = 12x^2$ if $y'(0) = 0$ and $y(0) = 0$

Solution:

$$y'' = 12x^2 \Rightarrow y' = 4x^3 + c_1$$
$$\Rightarrow y = x^4 + c_1 x + c_2$$

i.e. $y = x^4 + c_1 x + c_2$ is the general solution of the differential equation $y'' = 12x^2$

when $y'(0) = 0$ then we get the constant c_1

$$y'(0) = 0 + c_1 = 0 \Rightarrow c_1 = 0 \quad \Rightarrow y' = 4x^3$$

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When $y(0) = 0$

$$y = x^4 + c_1 x + c_2 \Rightarrow y(0) = 0 + 0 + c = 0 \Rightarrow c = 0$$
$$\therefore y = x^4$$

i.e. $y = x^4$ is the particular solution of the differential equation $y'' = 12x^2$

Exercise:

Q1\\ find the order and degree of all differential equations.

1) $y' = 8y$ 2) $y'^2 + xy' = y^2$
3) $\sqrt{y'' =} = 3y' + x$ 4) $y^{(4)} = \sqrt{y'}$
5) $(y'')^{\frac{1}{3}} = 6(1 + y'^2)^{\frac{5}{2}}$

Q2\\ prove the equation is solution of the differential equation responding

1) $y = x^2 + cx$ ....$xy' = x^2 + y$
2) $y = A\sin(2x) + B\cos(2x)$ ....$y'' + 4y = 0$
3) $y = Ae^{-x} + Be^{-2x}$ ....$y'' + 3y' + 2y = 0$
4) $y = Cx^2 + Bx + A$ ....$y'' = 0$
5) $y = Ae^x + Bx$...$y''(1-x) + y'x - y$

Differential equation of first order and first degree

A first order linear differential equation has the following form

$$\frac{dy}{dx} = f(x, y)$$
$$M(x, y)dx + N(x, y)dy = 0$$
$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Such that f, M, N function contain of x or y or both

Method solution of differential equation of first order and first degree

1- Variables are separable
2- Homogeneous equations
3- Exact equations

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4- Liner D.E. of order one

5- Bernoulli's equation

1- Variables are separable

A differential equation is called *separable* if it can be written as

$$f(y)dx + g(x)dy = 0$$

To solve a separable differential equation

1. Get all the y on the left hand side of the equation and all of the x on the right hand side.
2. Integrate both sides.
3. Plug in the given values to find the constant of integration (C)
4. Solve for y

$$f(y)dx + g(x)dy = 0 \Rightarrow \frac{dx}{g(x)} + \frac{dy}{f(y)} = 0$$

$$\int \frac{dx}{g(x)} + \int \frac{dy}{f(y)} = c$$

Example: Solve the differential equation $\frac{dy}{dx} = \frac{2y}{x}$

Solution:

$\frac{dy}{dx} = \frac{2y}{x} \Rightarrow xdy = 2ydx$ by divide (xy) the equation

$$\frac{dy}{y} = \frac{2dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{2dx}{x} \Rightarrow \ln y = 2\ln x + \ln c$$

$$\Rightarrow \ln y = \ln(cx^2) \quad \Rightarrow \quad y = cx^2$$

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Example: Solve the differential equation $(x+1)\frac{dy}{dx}=x^2(y^2+1)$

Solution:

$$(x+1)\frac{dy}{dx}=x^2(y^2+1)\Rightarrow(x+1)dy=x^2(y^2+1)dx$$

$$\Rightarrow\frac{dy}{(y^2+1)}=\frac{x^2dx}{x+1}\Rightarrow\int\frac{1}{y^2+1}dy=\int\frac{x^2}{x+1}dx$$

$$\Rightarrow\tan^{-1}(y)=\int\frac{x^2-1+1}{x+1}dx$$

$$\Rightarrow\tan^{-1}(y)=\int(x-1)dx+\int\frac{1}{x+1}dx$$

$$\Rightarrow\tan^{-1}(y)=\frac{x^2}{2}+\ln(x+1)+c\Rightarrow y=\tan[\frac{x^2}{2}+\ln(x+1)+c]$$

Example: Solve the differential equation $y'=\frac{x-xy^2}{x^2y-y}$

Solution:

$$y'=\frac{x-xy^2}{x^2y-y}\Rightarrow\frac{dy}{dx}=\frac{x(1-y^2)}{y(x^2-1)}\Rightarrow y(x^2-1)dy=x(1-y^2)dx$$

$$\Rightarrow\frac{y(x^2-1)}{(x^2-1)(1-y^2)}dy=\frac{x(1-y^2)}{(x^2-1)(1-y^2)}dx$$

$$\int\frac{y}{1-y^2}dy=\int\frac{x}{x^2-1}dx$$

$$-\frac{1}{2}\ln(1-y^2)=\frac{1}{2}\ln(x^2-1)+\ln c$$

$$\ln(1-y^2)^{\frac{-1}{2}}=\ln(x^2-1)^{\frac{1}{2}}+\ln c$$

$$(1-y^2)^{\frac{-1}{2}}=c(x^2-1)^{\frac{1}{2}}\Rightarrow1-y^2=\frac{c}{x^2-1}$$

$$y^2=1-\frac{c}{x^2-1}$$

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Example: Solve the differential equation $\sin^2(x)\cos y\,dx+\sin y\sec x\,dy=0$

Solution:

$$\sin^2(x)\cos y\,dx+\sin y\sec x\,dy=0$$

$$\frac{\sin^2(x)\cos y}{\cos y\sec x}dx+\frac{\sin y\sec x}{\cos y\sec x}dy=0$$

$$\int\frac{\sin^2(x)}{\sec x}dx+\int\frac{\sin y}{\cos y}dy=c$$

$$\int\sin^2(x)\cos x dx+\int\frac{\sin y}{\cos y}dy=c$$

$$\frac{\sin^3(x)}{3}-\ln\cos y=c$$

Exercise: Q1\\ Solve differential equations the following

1) $(4+x)y'=y^3$

2) $e^{y^2}dx+x^2ydy=0$

3) $\cos x\cos ydx+\sin x\sin ydy=0$

4) $e^x(y-1)dx+2(e^x+4)dy=0$

5) $y'=xy$

6) $x^2dx+y(x-1)dy=0$

Q2\\ find general solution of equation $xyy'=1+y^2$ and find particular solution if $y(2)=3$

2-Homogeneous equations of first order and first degree

A function $f(x,y)$ is said to be homogeneous of degree n if the equation

$$f(tx,ty)=t^n f(x,y)\quad,\quad t>0$$

Example: Show the function homogeneous and find degree

1) $f(x,y)=7x^2+8xy-9x^2$

2) $f(x,y)=x^3-2y^3 5xy^2$

3) $f(x,y)=9x^2-xy+2x$

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Solution:

1) $f(x,y)=7x^2+8xy-9x^2$

$f(tx,ty)=7(tx)^2+8(xt)(ty)-9(tx)^2$

$=7t^2x^2+8t^2xy-9t^2x^2$

$=t^2(7x^2+8xy-9x^2)$

$=t^2 f(x,y)$

$\therefore$ $f(x,y)$ is homogeneous and degree 2

2) $f(x,y)=x^3-2y^3+5y^2x$

$f(tx,ty)=(tx)^3-2(ty)^3+5(ty)^2(tx)$

$=t^3x^3-2t^3y^3+t^3y^2x$

$=t^3(x^3-2y^3+5y^2x)$

$=t^3 f(x,y)$

$\therefore$ $f(x,y)$ is homogeneous and degree 3

3) $f(x,y)=9x^2-xy+2x$

$f(tx,ty)=9(tx)^2-(tx)(ty)+2(tx)$

$=9t^2x^2-t^2xy+2tx$

$\therefore$ $f(x,y)$ is non homogeneous

Definition: The differential equation $M(x,y)dx+N(x,y)dy$ is said to be homogeneous if *M,N* are both homogeneous functions

Example:

1) $(x^2-xy+y^2)dx+xydy=0$ is homogeneous and degree 2

2) $x^4ydx+y^5dy=0$ is homogeneous and degree 5

Solution of homogeneous differential equation

The substitution $y=vx$ (and therefore $dy=vdx+xdv$) transforms a homogeneous equation into a separable one.

Example: Solve the differential equation $(x^2-xy+y^2)dx-xydy=0$

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Solution

Let $y = vx \quad \Rightarrow dy = vdx + xdv$

Substation in the equation

$$(x^2 - vx^2 + v^2x^2)dx - v^2x^2(vdx + xdv) = 0$$
$$(x^2 - vx^2 + v^2x^2)dx - v^2x^2dx - vx^3dv = 0$$
$$(x^2 - vx^2 + v^2x^2 - v^2x^2)dx - vx^3dv = 0$$
$$(x^2 - vx^2)dx - vx^3dv = 0$$
$$x^2(1 - v)dx - vx^3dv = 0$$
$$\frac{1}{x}dx - \frac{v}{1-v}dv = 0 \quad \Rightarrow \int \frac{1}{x}dx + \int \frac{v}{v-1}dv = \ln c$$
$$\ln x + \int \frac{v-1+1}{v-1}dv = \ln c$$
$$\ln x + \int (1 + \frac{1}{v-1})dv = \ln c$$
$$\ln x + v + \ln(v-1) = \ln c$$
$$\ln x(v-1) + v = \ln c \Rightarrow x(v-1)e^v = c$$

Since $y = vx \quad \Rightarrow v = \frac{y}{x}$ **then**

$$x(\frac{y}{x} - 1)e^{\frac{y}{x}} = c \Rightarrow (y - x)e^{\frac{y}{x}} = c$$

Example: Solve the differential equation

$$xdy - (y + \sqrt{x^2 - y^2})dx = 0$$

Solution:

The differential equation is homogenous and one degree

Let $y = vx \quad \Rightarrow dy = vdx + xdv$

$$x(xdv + vdx) - (vx + \sqrt{x^2 - x^2v^2})dx$$

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$$xvdx + x^2dv - vxdx - x\sqrt{1-v^2}\,dx = 0$$

$$x^2dv - x\sqrt{1-v^2}\,dx = 0$$

$$\frac{x^2}{x^2\sqrt{1-v^2}}dv - \frac{x\sqrt{1-v^2}}{x^2\sqrt{1-v^2}}dx = 0$$

$$\int \frac{1}{\sqrt{1-v^2}}dv - \int \frac{1}{x}dx = \ln c$$

$$\sin^{-1}(v) - \ln x = \ln c \Rightarrow \sin^{-1}(v) = \ln(cx)$$

$$\sin^{-1}(\frac{y}{x}) = \ln(cx)$$

Exercise: Solve the differential equations of the following

1) $(xy - y^2)dx - x^2dy = 0$
2) $(x^2 + y^2)dx - 2xydy = 0$
3) $x[1 + e^{\frac{y}{x}}]dy + (x - y)e^{\frac{y}{x}}dx = 0$
4) $(2xy + y^2)dx - 2x^2dy = 0$
5) $xy^2dy - (x^3 + y^3)dx = 0$
6) $xdy - (y + \sqrt{x^2 + y^2})dx$

2- Exact differential equations of first order and first degree

Definition: The differential equation $M(x,y)dx + N(x,y)dy$ is said to be exact if to find $f(x,y)$ be a function of two real variables such that F has continuous first partial derivatives

i.e. $df(x,y) = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = M(x,y)dx + N(x,y)dy$

The differential equation $M(x,y)dx + N(x,y)dy$ is exact if :

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Method of solution exact differential equations

When the differential equation is exact i.e. $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ the solution of following:

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$$\frac{\partial f(x,y)}{\partial x} = M(x,y) \quad .... \quad (1)$$

$$\frac{\partial f(x,y)}{\partial y} = N(x,y) \quad .... \quad (2)$$

1- Integration equation (1) with respect to x and y is constant and add integration of constant is function with respect to y i.e.

$$f(x,y) = \int M(x,y)dx + g(y) \quad .... (3)$$

2- derivative equation (3) with respect to y reduces :

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}[\int M(x,y)dx] + g'(x) \; .... (4)$$

3- equal equation (2) with (4) we find

$$N(x,y) = \frac{\partial}{\partial y}[\int M(x,y)dx] + g'(y)$$

For the equation find $g'(y)$ and integration with respect to y to find $g(y)$.

4- substation $g(y)$ in equation (3) we get the function $f(x,y)$

<u>Example</u>: Solve the differential equation

1) $(3x^2 + 3xy^2)dx + (3x^2y - 3y^2 + 2y)dy = 0$

2) $[\cos(2y) - 3x^2y^2]dx + [\cos(2y) - 2x\sin(2y) - 2x^3y]dy = 0$

3) $[x + y + 1]dx + [x - y^2 + 3]dy = 0$

<u>Solution</u>:

1) $(3x^2 + 3xy^2)dx + (3x^2y - 3y^2 + 2y)dy$

$M = 3x^2 + 3xy^2 \quad , \quad N = 3x^2y - 3y^2 + 2y$

$\frac{\partial M}{\partial y} = 6xy \quad , \quad \frac{\partial N}{\partial x} = 6xy$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

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$$f(x,y)=\int M(x,y)dx=\int(3x^2 3xy^2)dx$$

$$=x^3+\frac{3}{2}x^2y^2+g(y)$$

$$f(x,y)=x^3+\frac{3}{2}x^2y^2+g(y)$$

$$\frac{\partial f(x,y)}{\partial y}=3x^2y+g'(y)$$

$$\frac{\partial f(x,y)}{\partial y}=N=3x^2y+g'(y)$$

$$3x^2y-3y^2+2y=3x^2y+g'(y)\Rightarrow g'(y)=-3y^2+2y$$

$$g(y)=-y^3+y^2+c$$

$$\therefore\ f(x,y)=x^3+\frac{3}{2}x^2y^2-y^3+y^2+c$$

$2)[\cos(2y)-3x^2y^2]dx+[\cos(2y)-2x\sin(2y)-2x^3y]dy$

$M=\cos(2y)-3x^2y^2$ $\qquad$ $N=\cos(2y)-2x\sin(2y)-2x^3y$

$\frac{\partial M}{\partial y}=-2\sin(2y)-6x^2y$ $\qquad$ $\frac{\partial N}{\partial x}=-2\sin(2y)-6x^2y$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$f(x,y)=\int M(x,y)dx=\int[\cos(2y)-3x^2y^2]dx$$

$$f(x,y)=x\cos(2y)-x^3y^2+g(y)$$

$$\frac{\partial f}{\partial y}=-2x\sin(2y)-2x^3y+g'(y)$$

$$\frac{\partial f}{\partial y}=N\Rightarrow\quad -2x\sin(2y)-2x^3y+g'(y)=\cos(2y)-2x\sin(2y)-2x^3y$$

$$g'(y)=\cos(2y)\quad\Rightarrow\quad g(y)=\frac{1}{2}\sin(2y)+c$$

$$\therefore\ f(x,y)=x\cos(2y)-x^3y^2+\frac{1}{2}\sin(2y)+c$$

3) $(x+y+1)dx+(x-y^2+3)dy=0$

$M=x+y+1$ $\qquad$ $N=x-y^2+3$

$\frac{\partial M}{\partial y}=1$ $\qquad$ $\frac{\partial N}{\partial x}=1$ $\quad\Rightarrow\quad$ $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$f(x,y)=\int(x+y+1)dx\quad\Rightarrow f(x,y)=\frac{1}{2}x^2+xy+x+g(y)$$

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$$\frac{\partial M}{\partial y}=x+g'(y)$$

$$\frac{\partial M}{\partial y}=N \quad \Rightarrow \quad x+g'(y)=x-y^2+3 \quad \Rightarrow g'(y)=-y^2+3 \Rightarrow \quad g(y)=-\frac{1}{3}y^3+3y+c$$

Exercise:

$$f(x,y)=\frac{1}{2}x^2+xy+x-\frac{1}{3}y^3+3y+c$$

solve the differential equations of the following

1) $(3x^2-2y+e^{x+y})dx+(e^{x+y}-2x+4)dy=0$
2) $[\sin(2y)-2x\cos(2y)]dx+[2x\cos(2y)+2x^2\sin(2y)]dy=0$
3) $(x^3+xy^2-y)dx+(y^3+x^2y-x)dy$
4) $[x+\sin y-\cos y]dx+x[\sin y+\cos y]dy=0$
5) $e^{2x}dy+2ye^{2x}dx=x^2dx$

Integration Factorial

1-If a differential equation of the form $M(x,y)dx+N(x,y)dy$

When $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ **i.e. the differential equation is not exact .the integration factorial** R **is the following:**

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)\Rightarrow R=e^{\int f(x)dx}$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y)\Rightarrow R=e^{-\int g(y)dy}$$

multiplying both sides of $M(x,y)dx+N(x,y)dy$ **by** R **then become exact and we find of the solve .**

example: Solve the differential equation

$$(3xy^3+4y)dx+(3x^2y^2+2x)dy=0$$

Solution:

$M=3xy^3+4y \qquad\qquad N=3x^2y^2+2x$

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$$\frac{\partial M}{\partial y}=9xy^2+4 \qquad \frac{\partial N}{\partial x}=6xy^2+2$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \quad then$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{9xy^2+4-6xy^2+2}{3x^2y^2+2x}=\frac{3xy^2+2}{x(3xy^2+2)}=\frac{1}{x}$$

$$\therefore R=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

$$(3x^2y^3+4xy)dx+(3x^3y^2+2x^2)dy=0$$

$$\frac{\partial M}{\partial y}=9x^2y^2+4x \qquad \frac{\partial N}{\partial x}=9x^2y^2+4x$$

$$\therefore\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$f(x,y)=\int Mdx=\int(3x^2y^3+4xy)dx=x^3y^3+2x^2y+g(y)$$

$$\frac{\partial f}{\partial y}=3x^3y^2+2x^2+g'(y) \qquad \Rightarrow\frac{\partial f}{\partial y}=N$$

$$3x^3y^2+2x^2+g'(y)=3x^3y^2+2x^2\Rightarrow g'(y)=0\Rightarrow g(y)=c$$

$$\therefore f(x,y)=x^3y^3+2x^2y+c$$

Example: Solve the differential al equation

$$(2xy^2-2y)dx+(3x^2y-4x)dy=0$$

Solution:

$$M=2xy^2-2y \qquad N=3x^2y-4x$$

$$\frac{\partial M}{\partial y}=4xy-2 \qquad \frac{\partial N}{\partial x}=6xy-4$$

$$\therefore\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{4xy-2-6xy+4}{2xy^2-2y}=\frac{-2xy+2}{y(2xy\ \ -2)}=\frac{-(2xy-2)}{y(2xy-2)}=\frac{-1}{y}=g(y)$$

$$R=e^{-\int\frac{-1}{y}dy}=e^{\ln y}=y$$

$$\therefore(2xy^3-2y^2)dx+(3x^2y^2-4xy)dy=0$$

$$M=2xy^3-2y^2\Rightarrow\frac{\partial M}{\partial y}=6xy^2-4y \quad , N=3x^2y^2-4xy\Rightarrow\frac{\partial N}{\partial x}=6xy^2-4y$$

$\therefore x^2y^3-2xy^2=c$ **is exact of solution**

2- The differential equation is from $(pydx + qxdy)$ .such that p,q are real numbers then the integration factorial R :

$$R = x^{p-1}y^{q-1}$$

Example: Solve the differential equations.

1) $xdy - ydx = x^2y^3dx$
2) $xdy - 3ydx = x^4y^{-1}dx$
3) $xdy + ydx = xy^3dx$
4) $xdy - ydx = y^3(x^2 + y^2)dy$
5) $2ydx + 3xdy = 3x^{-1}dy$

Solution :

1) $xdy - ydx = x^2y^3dx$

$p = -1 \quad , q = 1$

$R = x^{p-1}y^{q-1} = x^{-1-1}y^{1-1} \quad \Rightarrow R = x^{-2}$

$x^{-1}dy - x^{-2}ydx = y^3dx$

$d(x^{-1}y) = y^3dx$

$Let \quad z = x^{-1}y \quad \Rightarrow \quad z = \dfrac{y}{x} \quad \Rightarrow y = xz$

$d(z) = (xz)^3dx \quad \Rightarrow dz = x^3z^3dx \quad \Rightarrow \dfrac{dz}{z^3} = x^3dx \Rightarrow z^{-3}dz = x^3dx$

$\dfrac{z^{-2}}{-2} = \dfrac{x^4}{4} + c \quad \Rightarrow \quad -\dfrac{1}{2z^2} = \dfrac{1}{4}x^4 + c$

$\sin ce \quad z = x^{-1}y \Rightarrow z = \dfrac{y}{x} \quad then$

$-\dfrac{1}{2(\frac{y}{x})^2} = \dfrac{1}{4}x^4 + c \quad \Rightarrow \quad -\dfrac{x^2}{2y^2} = \dfrac{1}{4}x^4 + +c$

2) $xdy - 3ydx = x^4y^{-1}dx$

$R = x^{p-1}y^{q-1} \quad \Rightarrow R = x^{-3-1}y^{1-1} \quad \Rightarrow R = x^{-4}$

$x^{-3}dy - 3x^{-4}ydx = y^{-1}dx$

$d(x^{-3}y) = y^{-1}dx$

$Let \quad z = x^{-3}y \Rightarrow z = \dfrac{y}{x^3} \Rightarrow y = x^3z$

$dz = (x^3z)^{-1}dx \quad \Rightarrow \quad zdz = x^{-3}dx \Rightarrow \quad \dfrac{1}{2}z^2 = -\dfrac{1}{2}x^{-2} + c$

$\dfrac{1}{2}(\dfrac{y}{x^3})^2 = -\dfrac{1}{2}x^{-2} + c \quad \Rightarrow \dfrac{y^2}{x^6} + \dfrac{1}{x^2} = c$

3) $xdy + ydx = xy^3dx$

$R = x^{p-1}y^{q-1} \Rightarrow R = x^{1-1}y^{1-1} \Rightarrow R = 1$

$\therefore xdy + ydx = xy^3dx$

$d(xy) = xy^3dx$

$Let\ z = xy \Rightarrow y = \frac{z}{x}$

$dz = x(\frac{z}{x})^3dx \Rightarrow dz = \frac{z^3}{x^2}dx \Rightarrow z^{-3}dz = x^{-2}dx \Rightarrow -\frac{1}{2z^2} = -\frac{1}{x} + c$

$\sin ce\ z = xy\ then$

$\frac{-1}{2(xy)^2} + \frac{1}{x} = c$

4) $xdy - ydx = y^3(x^2 + y^2)dy$

$R = x^{p-1}y^{q-1} \Rightarrow R = x^{-1-1}y^{1-1} \Rightarrow R = x^{-2}$

$x^{-1}dy - x^{-2}ydx = y^3(1 + x^{-2}y^2)dy$

$d(x^{-1}y) = y^3(1 + x^{-2}y^2)dy$

$Let\ z = x^{-1}y \Rightarrow y = xz$

$dz = y^3(1+z^2)dy \Rightarrow \int \frac{dz}{1+z^2} = \int y^3dy \Rightarrow \tan^{-1}(z) = \frac{1}{4}y^4 + c$

$\sin ce\ z = x^{-1}y\ then$

$\tan^{-1}(x^{-1}y) = \frac{1}{4}y^4 + c$

5) $2ydx + 3xdy = 3x^{-1}dy$

$R = x^{p-1}y^{q-1} \Rightarrow R = x^{2-1}y^{3-1} \Rightarrow R = xy^2$

$\therefore 2xy^3dx + 3x^2y^2dy = 3y^2dy$

$d(x^2y^3) = 3y^2dy$

$x^2y^3 = y^3 + c$

Exercise: solve the differential equations of the following:

1) $xdy + 3ydx = x^{-2}e^xdx$

2) $ydx + xdy = \sqrt{(x^2 + y^2)}\ (xdx + ydy)$

3) $3ydx + 4xdy = 5x^2y^{-3}dx$

4) $4ydx + xdy = xy^2dx$

5) $xdy - 2ydx = x^3y^4dy$

6) $xdy - ydx = (y^2 - 3)dy$

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